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Matrix Model and Refined Wall-Crossing Formula

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Abstract: In this paper, we show how to get matrix models corresponding to the refined BPS states partition functions of $\mathbb{C}^3$, resolved conifold and $\mathbb{C}^3/\mathbb{Z}_2$ by inserting the identity operator at a proper position in the fermionic expression of the refined BPS states partition functions.

Keywords: Matrix model, Refined BPS states partition function, different chambers, Vertex operators in 2d free fermions.

Contents

1. Introduction

Lately there has been progress in understanding the space of BPS states, $\mathcal{H}_{BPS}$, in type IIA string compactifications on Calabi-Yau threefolds. In general, such compactifications give rise to the effective $\mathcal{N} = 2$ theories in four dimensions. $\mathcal{H}_{BPS}$ is a special subspace of the full Hilbert space which is the one-particle representation of the $d = 4, \mathcal{N} = 2$ supersymmetry algebra. It contains lots of information about the Calabi-Yau threefold X and can be viewed as a bridge connecting the black hole physics and topological strings [1].

Due to the existence of the universal hypermultiplets, $\mathcal{H}_{BPS}(\gamma)$ has the following decomposition

$$\mathcal{H}_{BPS}(\gamma) = (\mathbf{0}, \mathbf{0}; \mathbf{\frac{1}{2}}) \otimes \mathcal{H}'_{BPS}(\gamma), \tag{1.1}$$

where γ is given by the generalized Mukai vector of the stable coherent sheaves corresponding to the D6/D4/D2/D0 branes

$$\begin{array}{rcccccccc} \gamma = ch(\mathcal{E})\sqrt{\hat{A}(X)} = & p^0 & + & P & + & Q & + & q_0 \\ \in & \mathrm{H}^0 & \oplus & \mathrm{H}^2 & \oplus & \mathrm{H}^4 & \oplus & \mathrm{H}^6 \\ & D6 & & D4 & & D2 & & D0 \end{array} \tag{1.2}$$

It is well known that the space $\mathcal{H}'_{BPS}$ depends on the asymptotic boundary conditions in the four-dimensional spacetime, where the boundary conditions in IIA compactification are the complexified Kähler moduli $u = iJ + B$ of the Calabi-Yau threefold X [2]. Roughly speaking, $\mathcal{H}'_{BPS}(\gamma, u) \sim \mathrm{H}^*(\mathcal{M}(\gamma, u))$, where $\mathcal{M}(\gamma, u)$ is the moduli space of stable coherent sheaves with the generalized Mukai vector γ under certain u-dependent stability condition [3]. The Spin(3) action on $\mathcal{H}_{BPS}$ gives rise to the following refined index of $\mathcal{H}'_{BPS}$ [4], after factorizing the contribution of the universal hypermultiplets,

$$\Omega^{ref}(\gamma, u, y) := \mathrm{Tr}_{\mathcal{H}'_{BPS}(\gamma,u)}(-y)^{2J'_3}, \tag{1.3}$$

where J'_3 is the reduced angular momentum [2]. $\Omega(\gamma, u, y)$ is conjectured to be related to the Poincaré polynomial of the BPS states moduli space [4]. Like the unrefined case we may define the refined BPS states partition function [4] by

$$Z^{ref}_{BPS}(q, Q, y, u) := \sum_{\substack{\beta \in \mathrm{H}_2(X;\mathbb{Z}) \\ n \in \mathbb{Z}}} (-q)^n Q^\beta \Omega^{ref}(\gamma_{\beta,n}, u, y). \tag{1.4}$$

In [5], we have shown how to use the vertex operators in 2d free fermions and the crystal corresponding to the Calabi-Yau threefold X to reproduce the wall-crossing formula of the refined BPS states partition function. In [5], we also conjecture that for the toric CY without any compact four-cycles we have the following formulas

$$Z^{ref}_{BPS}(q_1, q_2, Q)|_{chamber} = Z^{ref}_{top}(q_1, q_2, Q) Z^{ref}_{top}(q_1, q_2, Q^{-1})|_{chamber}. \tag{1.5}$$

In this paper, we present a connection between the matrix model with the Z^{ref}_{BPS} by employing the method in [6] to insert the identity operator at a proper position to get a one-matrix model corresponding to the refined BPS states partition function. In section 2 we review the work of [6]. In section 3 we show how to get the matrix model corresponding to the refined BPS states partition function. In section 4 we give the summary and discussion on future research directions.

2. Matrix model and wall-crossing formula

In this section we will review the matrix model for three dimensional toric Calabi-Yau geometry without any compact four-cycles arising from a triangulation of a strip [7, 6]. Let us denote the Euler characteristic of the Calabi-Yau as χ. Then the number of base $\mathbb{P}^1$ of a toric CY 3-fold will be $\chi - 1$ (see figure 1).

We may define the following creation and annihilation operators by using the vertex operator in the 2d free fermions [7, 8]:

$$A_-(x) := \prod_{i=1}^{\chi} \Gamma_-^{s_i} \left(x \prod_{j=0}^{i-1} q_j \right), \tag{2.1}$$

and

$$A_+(x) := \prod_{i=1}^{\chi} \Gamma_+^{s_i} \left(xq \prod_{j=0}^{i-1} q_j^{-1} \right), \tag{2.2}$$

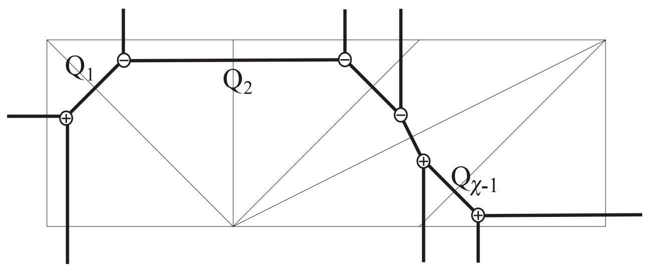

Figure 1: Toric diagram for Calabi-Yau threefold without compact four-cycles arises from a triangulation of a strip copied from [6].

where $s_i = 1$ or -1, and q is defined in terms of the eigenvalues q_i of all color operators as

$$q := \prod_{i=0}^{\chi-1} q_i. \tag{2.3}$$

The convention of Γ matrices we will use is

$$\Gamma_{\pm}^{s_i=+1}(x) = \Gamma_{\pm}(x), \qquad \Gamma_{\pm}^{s_i=-1}(x) = \Gamma'_{\pm}(x), \tag{2.4}$$

where the vertex operators Γ are derived from two dimensional free fermion theory [7, 8] and they satisfy the following commutation relation:

$$\Gamma_{+}^{s_1}(x)\Gamma_{-}^{s_2}(y) = (1 - s_1 s_2 xy)^{-s_1 s_2}\Gamma_{-}^{s_2}(y)\Gamma_{+}^{s_1}(x). \tag{2.5}$$

In terms of free fermions, the BPS partition functions can be expressed as correlation functions of the vertex operators in 2d free fermions [7]. The ket and bra states of the NCDT chamber are generated by the creation and annihilation operators as follows:

$$|\Omega_-\rangle := \prod_{r=0}^{\infty} A_-(q^r)|0\rangle, \qquad \langle\Omega_+| := \langle 0|\prod_{l=0}^{\infty} A_+(q^l). \tag{2.6}$$

Therefore the partition function for the NCDT chamber is

$$Z = \langle\Omega_+|\Omega_-\rangle\,. \tag{2.7}$$

The corresponding matrix model partition function is obtained by inserting the identity operator $\mathbb{I}$ of Hermitian matrix models, namely

$$Z_{\text{matrix}} = \langle\Omega_+|\mathbb{I}|\Omega_-\rangle, \tag{2.8}$$

where

$$\mathbb{I} = \int dU \left(\prod_{i=1}^{\infty} \Gamma'_{-}(u_i) \middle| 0 \right\rangle \left\langle 0 \middle| \prod_{j=1}^{\infty} \Gamma'_{+}(u_j^{-1}) \right). \tag{2.9}$$

Here dU is the unitary measure for $U(\infty)$ and $u_i = e^{i\phi_i}$ are the eigenvalues of U:

$$dU = \prod_k d\phi_i \prod_{i<j} (e^{i\phi_i} - e^{i\phi_j})(e^{-i\phi_i} - e^{-i\phi_j}). \tag{2.10}$$

2.1 Matrix model for $\mathbb{C}^3$

The toric diagram of $\mathbb{C}^3$ is shown in figure 2.

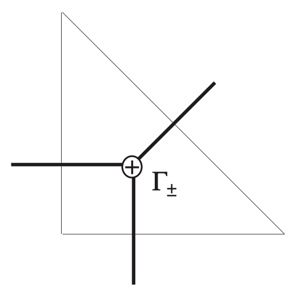

Figure 2: Toric diagram for $\mathbb{C}^3$.

According to [7], we may define $A_-(x) := \Gamma_-^s(x)$ and $A_+(x) := \Gamma_+^s(xq)$.[1] We can split the integrand of the matrix model partition function into

$$\left\langle 0 \middle| \prod_{l=0}^{\infty} A_+ \left(q^l\right) \prod_{i=1}^{\infty} \Gamma'_{-}(u_i) \middle| 0 \right\rangle = \prod_{i=1}^{\infty} \prod_{l=0}^{\infty} \left(1 + s u_i q^{l+1}\right)^s, \tag{2.11}$$

and

$$\left\langle 0 \middle| \prod_{j=1}^{\infty} \Gamma'_{+}(u_j^{-1}) \prod_{r=0}^{\infty} A_-(q^r) \middle| 0 \right\rangle = \prod_{j=1}^{\infty} \prod_{r=0}^{\infty} \left(1 + s u_j^{-1} q^r\right)^s. \tag{2.12}$$

Therefore the integrand of the matrix model is

$$\prod_{j=1}^{\infty} \prod_{r=0}^{\infty} \left(1 + s u_j^{-1} q^r\right)^s \prod_{i=1}^{\infty} \prod_{l=0}^{\infty} \left(1 + s u_i q^{l+1}\right)^s = \begin{cases} \mathrm{Det}\left(\prod_{r=0}^{\infty} \left(1 + U^{-1} q^r\right) \left(1 + U q^{r+1}\right) \right), & s = 1 \\ \mathrm{Det}^{-1}\left(\prod_{r=0}^{\infty} \left(1 - U^{-1} q^r\right) \left(1 - U q^{r+1}\right) \right), & s = -1 \end{cases}. \tag{2.13}$$

where $U \in U(\infty)$ whose eigenvalues are u_i.

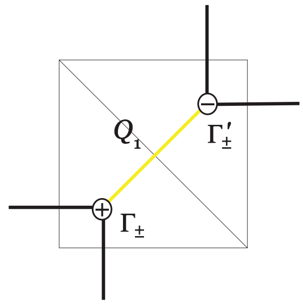

Figure 3: Toric diagram for the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{P}^1$.

2.2 Matrix model for the resolved conifold

Figure 3 is the toric diagram for the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{P}^1$. According to the discussion of the vertex on a strip [9], two $\mathbb{C}^3$ are connected by a $(-1,-1)$ curve, thus $s_2 = -s_1$. Therefore we can choose $s_1 = s$ and $s_2 = -s$. In the NCDT chamber we denote $q = q_0 q_1$, and $Q = -q_1$. Thus we can produce the BPS partition function by counting the pyramid model [10, 7, 11].

According to [7], in the NCDT chamber, the creation operator and the annihilation operator are defined as follows:

$$A_-(x) := \prod_{i=1}^{\chi} \Gamma_-^{s_i}\left(x \prod_{j=0}^{i-1} q_j\right) = \Gamma_-^{s}(xq_0)\,\Gamma_-^{-s}(xq), \tag{2.14}$$

and

$$A_+(x) := \prod_{i=1}^{\chi} \Gamma_+^{s_i}\left(xq \prod_{j=0}^{i-1} q_j^{-1}\right) = \Gamma_+^{s}(xq_1)\,\Gamma_+^{-s}(x). \tag{2.15}$$

After inserting the matrix identity $\mathbb{I}$ we get two matrix elements

$$\left\langle 0 \left| \prod_{j=1}^{\infty} \Gamma'_+\left(u_j^{-1}\right) \prod_{r=0}^{\infty} A_-\left(q^r\right) \right| 0 \right\rangle = \prod_{j=1}^{\infty}\prod_{r=0}^{\infty}\left(1 + s u_j^{-1} q^r q_0\right)^s \left(1 - s u_j^{-1} q^{r+1}\right)^{-s}, \tag{2.16}$$

and

$$\left\langle 0 \left| \prod_{l=0}^{\infty} A_+\left(q^l\right) \prod_{i=1}^{\infty} \Gamma'_-\left(u_i\right) \right| 0 \right\rangle = \prod_{i=1}^{\infty}\prod_{l=0}^{\infty}\left(1 - s u_i q^l\right)^{-s} \left(1 + s u_i q^l q_1\right)^{s}. \tag{2.17}$$

Then the partition function of the matrix model in the NCDT chamber is

$$\int dU \mathrm{Det}^s \left(\prod_{k=1}^{\infty} \frac{\left(1 - sU^{-1}q^{k+1}Q^{-1}\right)\left(1 - sUq^kQ\right)}{\left(1 - sU^{-1}q^{k+1}\right)\left(1 - sUq^k\right)}\right). \tag{2.18}$$

[1]Here s can be chosen to be either $+1$ or -1 and the final two results may be connected by an analytic continuation [6].

For chamber $R > 0$, $0 < n < B < n+1$ the wall crossing operator is defined in [7] as

$$\overline{W}_{p=1}(x) = \Gamma_-^s(x)\hat{Q}_1\Gamma_+^{-s}(x)\hat{Q}_0. \tag{2.19}$$

Therefore the partition function of the matrix model in the chamber $(R > 0, 0 < n < B < n+1)$ is

$$\begin{aligned} Z_{BPS}|_{chamber\ n} &= \langle \Omega_+ \left| \mathbb{I} \left(\overline{W}_1(1)\right)^n \right| \Omega_- \rangle \\ &= \int dU \left\langle \Omega_+ \left| \prod_{i=1}^{\infty} \Gamma'_-(u_i) \right| 0 \right\rangle \left\langle 0 \left| \prod_{j=1}^{\infty} \Gamma'_+\left(u_j^{-1}\right) \left(\overline{W}_1(1)\right)^n \right| \Omega_- \right\rangle . \end{aligned} \tag{2.20}$$

There are ambiguities of the position of the matrix identity operator which will result in different partition functions of the matrix model. We will discuss this problem in the final section.

We defined $\overline{W}'_{p=1}(x)$ by

$$\overline{W}'_{p=1}(x) := \Gamma_+^s(x)\hat{Q}_1\Gamma_-^{-s}(x)\hat{Q}_0. \tag{2.21}$$

In short we list the partition functions $Z_{n|p}$ for all chambers in the resolved conifold $(n \geq 0)$

$$\begin{array}{ll} R > 0, B \in [n, n+1] & Z_{n|1} = \langle \Omega_+ | \overline{W}_1^n | \Omega_- \rangle \ , \\ R > 0, B \in [-n-1, -n] & Z'_{n+1|1} = \langle \Omega_+ | (\overline{W}'_1)^{n+1} | \Omega_- \rangle \ , \\ R < 0, B \in [n, n+1] & \widetilde{Z}_{n+1|1} = \langle 0 | \overline{W}_1^{n+1} | 0 \rangle \ , \\ R < 0, B \in [-n-1, -n] & \widetilde{Z}'_{n|1} = \langle 0 | (\overline{W}'_1)^n | 0 \rangle \ . \end{array}$$

The corresponding matrix models are the results of the insertion of the identity operator in the partition functions respectively.

2.3 Matrix model for $\mathbb{C}^3/\mathbb{Z}_2$

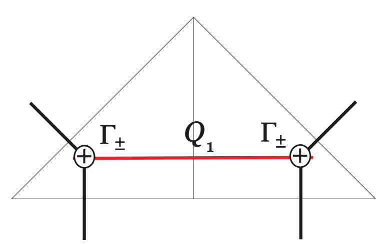

Figure 4: Toric diagram for the resolved $\mathbb{C}^3/\mathbb{Z}_2$.

Figure 4 is the toric diagram of $\mathcal{O}_{\mathbb{P}^1}(-2,0)$ which is the resolved $\mathbb{C}^3/\mathbb{Z}_2$. The vertex strip in this case is different from the resolved conifold by $s_1 = s_2$ rather than $s_1 = -s_2$. In the NCDT chamber, we define the creation operator

$$A_-(x) := \prod_{i=1}^{\chi} \Gamma_-^{s_i}\left(x \prod_{j=0}^{i-1} q_j\right) = \Gamma_-^s(xq_0)\,\Gamma_-^s(xq), \tag{2.22}$$

and the annihilation operator

$$A_{+}(x):=\prod_{i=1}^{\chi}\Gamma_{+}^{s_i}\left(xq\prod_{j=0}^{i-1}q_j^{-1}\right)=\Gamma_{+}^{s}\left(xq_1\right)\Gamma_{+}^{s}(x). \tag{2.23}$$

The insertion of the matrix identity $\mathbb{I}$ results in two matrix elements

$$\left\langle 0\left|\prod_{j=1}^{\infty}\Gamma_{+}'\left(u_j^{-1}\right)\prod_{r=0}^{\infty}A_{-}\left(q^{r}\right)\right|0\right\rangle=\prod_{j=1}^{\infty}\prod_{r=0}^{\infty}\left(1+su_j^{-1}q^{r}q_0\right)^{s}\left(1+su_j^{-1}q^{r+1}\right)^{s}, \tag{2.24}$$

and

$$\left\langle 0\left|\prod_{l=0}^{\infty}A_{+}\left(q^{l}\right)\prod_{i=1}^{\infty}\Gamma_{-}'\left(u_i\right)\right|0\right\rangle=\prod_{i=1}^{\infty}\prod_{l=0}^{\infty}\left(1+su_iq^{l}\right)^{s}\left(1+su_iq^{l}q_1\right)^{s}. \tag{2.25}$$

Then the partition function of the matrix model is

$$\int dU\mathrm{Det}^{s}\left(\prod_{k=1}^{\infty}\left(1-sU^{-1}q^{k+1}Q^{-1}\right)\left(1-sUq^{k}Q\right)\left(1+sU^{-1}q^{k+1}\right)\left(1+sUq^{k}\right)\right). \tag{2.26}$$

The wall crossing operator is defined in [7] as

$$\overline{W}_1(x)=\Gamma_{+}^{s}(x)\hat{Q}_1\Gamma_{-}^{s}(x)\hat{Q}_0. \tag{2.27}$$

Therefore the partition function of the matrix model for chamber $R>0$, $0<n<B<n+1$ is

$$Z_{BPS}|_{chamber\ n}=\int dU\left\langle\Omega_{+}\left|\prod_{i=1}^{N}\Gamma_{-}'\left(u_i\right)\right|0\right\rangle\left\langle 0\left|\prod_{j=1}^{N}\Gamma_{+}'\left(u_j^{-1}\right)\left(\overline{W}_1(1)\right)^{n}\right|\Omega_{-}\right\rangle. \tag{2.28}$$

Similarly we can get the matrix models in all the chambers as previous section.

3. refined Matrix model and refined wall-crossing formula

In this section, we will use techniques introduced in previous section and the refined BPS states partition functions proposed in [5] to obtain refined matrix model for several typical toric Calabi-Yau 3-folds.

3.1 Refined matrix model for $\mathbb{C}^3$

In [5], we define the creation and annihilation operators as

$$\overline{A}_{-}(x):=\hat{Q}_{0,-}^{\frac{1}{2}-\frac{\delta}{2}}\Gamma_{-}(x)\hat{Q}_{0,-}^{\frac{1}{2}+\frac{\delta}{2}}=\Gamma_{-}\left(xq_2^{\frac{1}{2}-\frac{\delta}{2}}\right)\hat{Q}_{0,-}, \tag{3.1}$$

$$\overline{A}_{+}(x):=\hat{Q}_{0,+}^{\frac{1}{2}-\frac{\delta}{2}}\Gamma_{+}(x)\hat{Q}_{0,+}^{\frac{1}{2}+\frac{\delta}{2}}=\hat{Q}_{0,+}\Gamma_{+}\left(xq_1^{\frac{1}{2}+\frac{\delta}{2}}\right), \tag{3.2}$$

and states

$$\langle \Omega_+| := \langle 0|\overline{A}_+(1)\cdots\overline{A}_+(1) = \langle 0|\prod_{i=1}^{\infty}\Gamma_+(q_1^{i-\frac{1}{2}+\frac{\delta}{2}}), \tag{3.3}$$

$$|\Omega_-\rangle := \overline{A}_-(1)\cdots\overline{A}_-(1)|0\rangle = \prod_{j=1}^{\infty}\Gamma_-(q_2^{j-\frac{1}{2}-\frac{\delta}{2}})|0\rangle. \tag{3.4}$$

In the general convention, we can rewrite $\langle\Omega_+^s|$ and $|\Omega_-^s\rangle$ as follows

$$\langle\Omega_+^s| := \langle 0|\prod_{i=1}^{\infty}\Gamma_+^s(q_1^{i-\frac{1}{2}+\frac{\delta}{2}}) = \begin{cases} \langle 0|\prod_{i=1}^{\infty}\Gamma_+(q_1^{i-\frac{1}{2}+\frac{\delta}{2}}) & \text{if } s=1, \\ \langle 0|\prod_{i=1}^{\infty}\Gamma'_+(q_1^{i-\frac{1}{2}+\frac{\delta}{2}}) & \text{if } s=-1, \end{cases} \tag{3.5}$$

$$|\Omega_-^s\rangle := \prod_{j=1}^{\infty}\Gamma_-^s(q_2^{j-\frac{1}{2}-\frac{\delta}{2}})|0\rangle = \begin{cases} \prod_{j=1}^{\infty}\Gamma_-(q_2^{j-\frac{1}{2}-\frac{\delta}{2}})|0\rangle & \text{if } s=1, \\ \prod_{j=1}^{\infty}\Gamma'_-(q_2^{j-\frac{1}{2}-\frac{\delta}{2}})|0\rangle & \text{if } s=-1. \end{cases} \tag{3.6}$$

Then the refined BPS states partition function is

$$\mathcal{Z}_{BPS}^{ref} = \langle\Omega_+^s|\Omega_-^s\rangle = M_\delta(q_1,q_2) \tag{3.7}$$

where the refined MacMahon function $M_\delta(q_1,q_2)$ is defined by

$$M_\delta(q_1,q_2) = \prod_{i,j=1}^{\infty}(1-q_1^{i-\frac{1}{2}+\frac{\delta}{2}}q_2^{j-\frac{1}{2}-\frac{\delta}{2}})^{-1}. \tag{3.8}$$

In order to get a matrix model we insert the identity operator $\mathbb{I}$ into the formula (3.7)

$$\mathcal{Z}_{BPS}^{ref} = \left\langle 0\left|\prod_{i=1}^{\infty}\Gamma_+^s(q_1^{i-\frac{i}{2}+\frac{\delta}{2}})\ \mathbb{I}\ \prod_{j=1}^{\infty}\Gamma_-^s(q_2^{j-\frac{1}{2}-\frac{\delta}{2}})\right|0\right\rangle = M_\delta(q_1,q_2). \tag{3.9}$$

Due to the formula (2.9) of $\mathbb{I}$, the matrix elements are

$$\begin{aligned}\left\langle 0\left|\prod_{j=1}^{\infty}\Gamma'_+\left(u_j^{-1}\right)\right|\Omega_-^s\right\rangle &= \left\langle 0\left|\prod_{j=1}^{\infty}\Gamma'_+\left(u_j^{-1}\right)\prod_{k=1}^{\infty}\Gamma_-^s\left(q_2^{k-\frac{1}{2}-\frac{\delta}{2}}\right)\right|0\right\rangle \\ &= \prod_{k=1}^{\infty}\prod_{j=1}^{\infty}\left(1+su_j^{-1}q_2^{k-\frac{1}{2}-\frac{\delta}{2}}\right)^s, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\left\langle \Omega_+^s\left|\prod_{i=1}^{\infty}\Gamma'_-(u_i)\right|0\right\rangle &= \left\langle 0\left|\prod_{l=1}^{\infty}\Gamma_+^s\left(q_1^{l-\frac{1}{2}+\frac{\delta}{2}}\right)\prod_{i=1}^{\infty}\Gamma'_-(u_i)\right|0\right\rangle \\ &= \prod_{l=1}^{\infty}\prod_{i=1}^{\infty}\left(1+su_iq_1^{l-\frac{1}{2}+\frac{\delta}{2}}\right)^s. \end{aligned} \tag{3.11}$$

Therefore the matrix model integrand is

$$\mathrm{Det}^s\left[\prod_{k=1}^{\infty}\left(1+sU^{-1}q_2^{k-\frac{1}{2}-\frac{\delta}{2}}\right)\left(1+sUq_1^{k-\frac{1}{2}+\frac{\delta}{2}}\right)\right]. \tag{3.12}$$

Finally we have

$$\mathcal{Z}_{BPS}^{ref} = \int dU \mathrm{Det}^s \left[\prod_{k=1}^{\infty}\left(1 + sU^{-1}q_2^{k-\frac{1}{2}-\frac{\delta}{2}}\right)\left(1 + sUq_1^{k-\frac{1}{2}+\frac{\delta}{2}}\right)\right], \tag{3.13}$$

where dU denotes the unitary measure for $U(\infty)$. It is given by

$$dU = \prod_k d\phi_k \prod_{i<j}(e^{i\phi_i} - e^{i\phi_j})(e^{-i\phi_i} - e^{-i\phi_j}), \tag{3.14}$$

where $u_i = e^{i\phi_i}$ are the eigenvalues of U.

3.2 Refined matrix model for the resolved conifold

We follow the same logic as the $\mathbb{C}^3$ case. First we define $\langle\Omega_+^s|$ and $|\Omega_-^s\rangle$ by

$$|\Omega_-^s\rangle := \prod_{j=1}^{\infty}\Gamma_-^s\left(q_2^{j-\frac{1}{2}}(-Q)^{-\frac{1}{2}}\right)\Gamma_-^{-s}\left(q_2^{j-\frac{1}{2}}(-Q)^{\frac{1}{2}}\right)|0\rangle, \tag{3.15}$$

$$\langle\Omega_+^s| := \langle 0|\prod_{i=1}^{\infty}\Gamma_+^s\left(q_1^{i-\frac{1}{2}}(-Q)^{\frac{1}{2}}\right)\Gamma_+^{-s}\left(q_1^{i-\frac{1}{2}}(-Q)^{-\frac{1}{2}}\right). \tag{3.16}$$

Then the refined BPS states partition function in the NCDT chamber can be written as

$$\begin{aligned}\mathcal{Z}_{BPS}^{ref}\Big|_{NCDT} &= \langle\Omega_+^s|\Omega_-^s\rangle \\ &= (M_{\delta=0}(q_1,q_2))^2\prod_{i,j=1}^{\infty}(1 - q_1^{i-\frac{1}{2}}q_2^{j-\frac{1}{2}}Q)(1 - q_1^{i-\frac{1}{2}}q_2^{j-\frac{1}{2}}Q^{-1}).\end{aligned} \tag{3.17}$$

Actually there are some degrees of freedom of the variables of Γ operators in the formulas (3.15, 3.16). We find that in order to preserve the equation (3.17), the general definition of $\langle\Omega_+^s|$ and $|\Omega_-^s\rangle$ will be as follows

$$|\Omega_-^{(s,\delta_1,\delta_2)}\rangle := \prod_{j=1}^{\infty}\Gamma_-^s\left(q_2^{j-\frac{1}{2}+\delta_2}q_1^{\delta_1}(-Q)^{-\frac{1}{2}}\right)\Gamma_-^{-s}\left(q_2^{j-\frac{1}{2}+\delta_2}q_1^{\delta_1}(-Q)^{\frac{1}{2}}\right)|0\rangle, \tag{3.18}$$

$$\langle\Omega_+^{(s,\delta_1,\delta_2)}| := \langle 0|\prod_{i=1}^{\infty}\Gamma_+^s\left(q_1^{i-\frac{1}{2}-\delta_1}q_2^{-\delta_2}(-Q)^{\frac{1}{2}}\right)\Gamma_+^{-s}\left(q_1^{i-\frac{1}{2}-\delta_1}q_2^{-\delta_2}(-Q)^{-\frac{1}{2}}\right), \tag{3.19}$$

where δ_1, δ_2 are two arbitrary integers. Now we insert the identity operator $\mathbb{I}$ as follows:

$$\begin{aligned}\mathcal{Z}_{BPS}^{ref}\Big|_{NCDT} &= \langle\Omega_+^{(s,\delta_1,\delta_2)}|\mathbb{I}|\Omega_-^{(s,\delta_1,\delta_2)}\rangle = \langle 0|\prod_{i=1}^{\infty}\Gamma_+^s\left(q_1^{i-\frac{1}{2}-\delta_1}q_2^{-\delta_2}(-Q)^{\frac{1}{2}}\right)\Gamma_+^{-s}\left(q_1^{i-\frac{1}{2}-\delta_1}q_2^{-\delta_2}(-Q)^{-\frac{1}{2}}\right)\cdot \\ &\quad \mathbb{I}\prod_{j=1}^{\infty}\Gamma_-^s\left(q_2^{j-\frac{1}{2}+\delta_2}q_1^{\delta_1}(-Q)^{-\frac{1}{2}}\right)\Gamma_-^{-s}\left(q_2^{j-\frac{1}{2}+\delta_2}q_1^{\delta_1}(-Q)^{\frac{1}{2}}\right)|0\rangle.\end{aligned} \tag{3.20}$$

Thus we may obtain the following matrix elements

$$\begin{aligned}&\left\langle 0\left|\prod_{j=1}^{\infty}\Gamma'_+\left(u_j^{-1}\right)\prod_{r=1}^{\infty}\Gamma_-^s\left(q_2^{r-\frac{1}{2}+\delta_2}q_1^{\delta_1}(-Q)^{-\frac{1}{2}}\right)\Gamma_-^{-s}\left(q_2^{r-\frac{1}{2}+\delta_2}q_1^{\delta_1}(-Q)^{\frac{1}{2}}\right)\right|0\right\rangle \\ &= \prod_{j=1}^{\infty}\prod_{r=1}^{\infty}\left(1 + su_j^{-1}(-Q)^{-\frac{1}{2}}q_2^{r-\frac{1}{2}+\delta_2}q_1^{\delta_1}\right)^s\left(1 - su_j^{-1}(-Q)^{\frac{1}{2}}q_2^{r-\frac{1}{2}+\delta_2}q_1^{\delta_1}\right)^{-s},\end{aligned} \tag{3.21}$$

and

$$\left\langle 0 \left| \prod_{l=1}^{\infty} \Gamma_{+}^{s}\left(q_1^{l-\frac{1}{2}-\delta_1} q_2^{-\delta_2}(-Q)^{\frac{1}{2}}\right) \Gamma_{+}^{-s}\left(q_1^{l-\frac{1}{2}-\delta_1} q_2^{-\delta_2}(-Q)^{-\frac{1}{2}}\right) \prod_{i=1}^{\infty} \Gamma_{-}'(u_i) \right| 0 \right\rangle$$
$$= \prod_{i=1}^{\infty}\prod_{l=1}^{\infty}\left(1+su_i(-Q)^{\frac{1}{2}} q_1^{l-\frac{1}{2}-\delta_1} q_2^{-\delta_2}\right)^{s}\left(1-su_i(-Q)^{-\frac{1}{2}} q_1^{l-\frac{1}{2}-\delta_1} q_2^{-\delta_2}\right)^{-s}. \quad (3.22)$$

Then the partition of the matrix model is

$$Z_{BPS}^{ref}\Big|_{NCDT} = \int dU \mathrm{Det}^{s}\left[\prod_{k=0}^{\infty} \frac{\left(1+sU^{-1}(-Q)^{-\frac{1}{2}} q_2^{k-\frac{1}{2}+\delta_2} q_1^{\delta_1}\right)\left(1+sU(-Q)^{\frac{1}{2}} q_1^{k-\frac{1}{2}-\delta_1} q_2^{-\delta_2}\right)}{\left(1-sU^{-1}(-Q)^{\frac{1}{2}} q_2^{k-\frac{1}{2}+\delta_2} q_1^{\delta_1}\right)\left(1-sU(-Q)^{-\frac{1}{2}} q_1^{k-\frac{1}{2}-\delta_1} q_2^{-\delta_2}\right)}\right]. \quad (3.23)$$

For chamber $(R>0, 0<n<B<n+1)$, the partition function is

$$\begin{aligned} Z_{BPS}^{ref}\Big|_{(R>0,0<n<B<n+1)} = & \left\langle 0 \left| \prod_{k=1}^{\infty} \Gamma_{+}^{s}\left(q_1^{k-\frac{1}{2}-\delta_1} q_2^{-\delta_2}(-Q)^{\frac{1}{2}}\right) \Gamma_{+}^{-s}\left(q_1^{k+n-\frac{1}{2}-\delta_1} q_2^{-\delta_2}(-Q)^{-\frac{1}{2}}\right)\right.\right. \\ & \bullet \prod_{l=1}^{n} \Gamma_{-}^{s}\left(q_2^{l-\frac{1}{2}+\delta_2} q_1^{\delta_1}(-Q)^{-\frac{1}{2}}\right) \Gamma_{+}^{-s}\left(q_1^{n-l+\frac{1}{2}-\delta_1} q_2^{-\delta_2}(-Q)^{-\frac{1}{2}}\right) \\ & \bullet \left.\left. \prod_{k=1}^{\infty} \Gamma_{-}^{s}\left(q_2^{k+n-\frac{1}{2}+\delta_2} q_1^{\delta_1}(-Q)^{-\frac{1}{2}}\right) \Gamma_{-}^{-s}\left(q_2^{k-\frac{1}{2}+\delta_2} q_1^{\delta_1}(-Q)^{\frac{1}{2}}\right) \right| 0 \right\rangle . \end{aligned} \quad (3.24)$$

Now we insert the matrix identity operator $\mathbb{I}$ in the partition function and we show the details in appendix (A), then it gives rise to

$$\begin{aligned} Z_{BPS}^{ref}\Big|_{(R>0,0<n<B<n+1)} = & \int dU \prod_{l=1}^{n}\prod_{p=1}^{l}\left(1-Q^{-1} q_1^{n-l+\frac{1}{2}} q_2^{p-\frac{1}{2}}\right)^{-1} \\ \bullet \mathrm{Det}^{s} & \left[\prod_{k=0}^{\infty} \frac{\left(1+sU^{-1}(-Q)^{-\frac{1}{2}} q_2^{r-\frac{1}{2}+\delta_2} q_1^{\delta_1}\right)\left(1+sU(-Q)^{\frac{1}{2}} q_1^{l-\frac{1}{2}-\delta_1} q_2^{-\delta_2}\right)}{\left(1-sU^{-1}(-Q)^{\frac{1}{2}} q_2^{r-\frac{1}{2}+\delta_2} q_1^{\delta_1}\right)\left(1-sU(-Q)^{-\frac{1}{2}} q_1^{l-\frac{1}{2}-\delta_1} q_2^{-\delta_2}\right)}\right]. \end{aligned} \quad (3.25)$$

Similarly, the partition function of the matrix model for the chamber $(R>0, n-1<0<n\leq 0)$ is

$$\begin{aligned} Z_{BPS}^{ref}\Big|_{(R>0,n-1<B<n\leq 0)} = & \int dU \prod_{l=1}^{n}\prod_{p=1}^{l}\left(1-Q q_1^{n-l+\frac{1}{2}} q_2^{p-\frac{1}{2}}\right)^{-1} \\ \bullet \mathrm{Det}^{s} & \left[\prod_{k=0}^{\infty} \frac{\left(1+sU^{-1}(-Q)^{\frac{1}{2}} q_2^{r-\frac{1}{2}+\delta_2} q_1^{\delta_1}\right)\left(1+sU(-Q)^{-\frac{1}{2}} q_1^{l-\frac{1}{2}-\delta_1} q_2^{-\delta_2}\right)}{\left(1-sU^{-1}(-Q)^{-\frac{1}{2}} q_2^{r-\frac{1}{2}+\delta_2} q_1^{\delta_1}\right)\left(1-sU(-Q)^{\frac{1}{2}} q_1^{l-\frac{1}{2}-\delta_1} q_2^{-\delta_2}\right)}\right]. \end{aligned} \quad (3.26)$$

3.3 Refined matrix model for $\mathbb{C}^3/\mathbb{Z}_2$

According to [5], the refined BPS states partition function of $\mathbb{C}^3/\mathbb{Z}_2$ in the chamber $(R > 0, 0 \leq n < B < n+1)$ is

$$
\begin{aligned}
& Z^{ref}_{BPS}\Big|_{(R>0,0\leq n<B<n+1)} \\
&= \langle 0| \prod_{k=1}^{\infty} \Gamma^s_+ \left[q_1^{k+\delta_1} q_2^{\delta_2} (-Q)^{\frac{1}{2}}\right] \Gamma^s_+ \left[q_1^{k+n+\delta_1} q_2^{\delta_2} (-Q)^{-\frac{1}{2}}\right] \Gamma^s_- \left[q_2^{-\delta_2} q_1^{-\delta_1} (-Q)^{-\frac{1}{2}}\right] \cdot \\
&\quad \times \Gamma^s_+ \left[q_1^{n+\delta_1} q_2^{\delta_2} (-Q)^{-\frac{1}{2}}\right] \Gamma^s_- \left[q_2^{1-\delta_2} q_1^{-\delta_1} (-Q)^{-\frac{1}{2}}\right] \Gamma^s_+ \left[q_1^{n-1+\delta_1} q_2^{\delta_2} (-Q)^{-\frac{1}{2}}\right] \cdots \\
&\quad \times \Gamma^s_- \left[q_2^{n-1-\delta_2} q_1^{-\delta_1} (-Q)^{-\frac{1}{2}}\right] \Gamma^s_+ \left[q_1^{1+\delta_1} q_2^{\delta_2} (-Q)^{-\frac{1}{2}}\right] \cdot \\
&\quad \times \prod_{k=1}^{\infty} \Gamma^s_- \left[q_2^{k+n-1-\delta_2} q_1^{-\delta_1} (-Q)^{-\frac{1}{2}}\right] \Gamma^s_- \left[q_2^{k-1-\delta_2} q_1^{-\delta_1} (-Q)^{\frac{1}{2}}\right] |0\rangle \\
&= \prod_{l+r\leq n+1} (1 - q_1^l q_2^{r-1} Q^{-1}) \cdot \langle 0| \prod_{k=1}^{\infty} \Gamma^s_+ \left[q_1^{k+\delta_1} q_2^{\delta_2} (-Q)^{\frac{1}{2}}\right] \Gamma^s_+ \left[q_1^{k+\delta_1} q_2^{\delta_2} (-Q)^{-\frac{1}{2}}\right] \cdot \\
&\quad \times \prod_{k=1}^{\infty} \Gamma^s_- \left[q_2^{k-1-\delta_2} q_1^{-\delta_1} (-Q)^{-\frac{1}{2}}\right] \Gamma^s_- \left[q_2^{k-1-\delta_2} q_1^{-\delta_1} (-Q)^{\frac{1}{2}}\right] |0\rangle \\
&= M^2_{\delta=\frac{1}{2}}(q_1, q_2) \prod_{i,j=1}^{\infty} (1 - q_1^i q_2^{j-1} Q)^{-1} \prod_{i+j>n+1} (1 - q_1^i q_2^{j-1} Q^{-1})^{-1}
\end{aligned}
\tag{3.27}
$$

Now we insert the identity operator $\mathbb{I}$ as follows:

$$
\begin{aligned}
& \langle 0| \prod_{k=1}^{\infty} \Gamma^s_+ \left[q_1^{k+\delta_1} q_2^{\delta_2} (-Q)^{\frac{1}{2}}\right] \Gamma^s_+ \left[q_1^{k+\delta_1} q_2^{\delta_2} (-Q)^{-\frac{1}{2}}\right] \cdot \mathbb{I} \\
& \prod_{k=1}^{\infty} \Gamma^s_- \left[q_2^{k-1-\delta_2} q_1^{-\delta_1} (-Q)^{-\frac{1}{2}}\right] \Gamma^s_- \left[q_2^{k-1-\delta_2} q_1^{-\delta_1} (-Q)^{\frac{1}{2}}\right] |0\rangle.
\end{aligned}
\tag{3.28}
$$

Then the matrix elements are

$$
\begin{aligned}
& \left\langle 0 \left| \prod_{k=1}^{\infty} \Gamma^s_+ \left[q_1^{k+\delta_1} q_2^{\delta_2} (-Q)^{\frac{1}{2}}\right] \Gamma^s_+ \left[q_1^{k+\delta_1} q_2^{\delta_2} (-Q)^{-\frac{1}{2}}\right] \prod_{i=1}^{\infty} \Gamma'_-(u_i) \right| 0 \right\rangle \\
&= \prod_{i=1}^{\infty} \prod_{k=1}^{\infty} \left(1 + s u_i (-Q)^{-\frac{1}{2}} q_1^{k+\delta_1} q_2^{\delta_2}\right)^s \left(1 + s u_i (-Q)^{\frac{1}{2}} q_1^{k+\delta_1} q_2^{\delta_2}\right)^s,
\end{aligned}
\tag{3.29}
$$

and

$$
\begin{aligned}
& \left\langle 0 \left| \prod_{j=1}^{\infty} \Gamma'_+ \left(u_j^{-1}\right) \prod_{k=1}^{\infty} \Gamma^s_- \left[q_2^{k-1-\delta_2} q_1^{-\delta_1} (-Q)^{-\frac{1}{2}}\right] \Gamma^s_- \left[q_2^{k-1-\delta_2} q_1^{-\delta_1} (-Q)^{\frac{1}{2}}\right] \right| 0 \right\rangle \\
&= \prod_{j=1}^{\infty} \prod_{k=1}^{\infty} \left(1 + s u_j^{-1} (-Q)^{-\frac{1}{2}} q_2^{k-1-\delta_2} q_1^{-\delta_1}\right)^s \left(1 + s u_j^{-1} (-Q)^{\frac{1}{2}} q_2^{k-1-\delta_2} q_1^{-\delta_1}\right)^s.
\end{aligned}
\tag{3.30}
$$

Therefore the partition function of the matrix model for the $(R > 0, 0 \leq n < B < n+1)$ chamber is

$$
\begin{aligned}
& Z^{ref}_{BPS}\Big|_{(R>0,0\leq n<B<n+1)} \\
&= \prod_{l+r\leq n+1} (1 - q_1^l q_2^{r-1} Q^{-1})^{-1} \int dU \mathrm{Det}^s \left[\prod_{k=1}^{\infty} \left(1 + sU(-Q)^{-\frac{1}{2}} q_1^{k+\delta_1} q_2^{\delta_2}\right) \left(1 + sU(-Q)^{\frac{1}{2}} q_1^{k+\delta_1} q_2^{\delta_2}\right) \right. \\
& \quad \left. \times \left(1 + sU^{-1}(-Q)^{-\frac{1}{2}} q_2^{k-1-\delta_2} q_1^{-\delta_1}\right) \left(1 + sU^{-1}(-Q)^{\frac{1}{2}} q_2^{k-1-\delta_2} q_1^{-\delta_1}\right) \right].
\end{aligned}
\tag{3.31}
$$

4. Conclusion and discussion

In this paper, we use the free fermion version of refined BPS states partition functions to obtain their corresponding matrix models. But there still are some subtle problems hidden in calculations.

The first one is the choice of s, the "type" of the first vertex in a strip. As argued in [6], the final results should have an analytic continuation. Here we want to give some simple arguments on this analytic continuation. The key formula in this paper is the equation (2.5), from which we may see if we change one of the s's into its opposite one, then variables will go from numerator to denominator or from denominator to numerator. This is similar as the analytic continuation on $\mathbb{P}^1$, which has two patches and we can construct the analytic continuation from one patch to the other.

Another subtle problem is the choice of δ in the refined MacMahon function $M_\delta(q_1, q_2)$. In fact δ is an arbitrary constant set up by hand if we just want to get the generating function of 3d partition function. While it is not clear to us whether the choice of δ in the paper [12] is unique or not, how to get the refined MacMahon function appearing in [13] which gives the mathematical rigid refined BPS partition functions for the D0 branes, and whether those different refined MacMahon functions in [13] and [5] are physically identical.

The third subtle problem comes from the position of insertion of the identity operator. Apparently, a different inserting position will give rise to a different action of matrix model. But just as in QFT, an identity operator means summation over complete set of intermediate states, and inserting an identity operator at different positions just means we observe different stages of interactions. Actually, if we want to get a multi-matrix model rather than a one-matrix model we may insert more identity operators in the corresponding correlation function of refined BPS partition function.

In [6] besides getting the matrix models corresponding to the BPS partition function, the authors also find the following interesting property of the BPS partition function: the matrix model for the BPS counting on the CY X is related to the topological string partition function for another CY Y, whose Kähler moduli space $\mathcal{M}(Y)$ contains two copies of $\mathcal{M}(X)$, e.g. the partition function of matrix model corresponding to the BPS partition function on the conifold will be related to the topological string partition function on the SPP geometry. It would be interesting to see if the matrix model proposed in this paper

is related to the refined topological string partition function on another CY. This work is under consideration.

Along the line of techniques discussed in the paper, we can also obtain refined matrix models for any strip like toric CY quickly, and what's more, if we insert an identity in the equation (150) of [12] we can get a matrix model for the refined topological vertex. Since the refined topological vertex is the element to generate 5d instanton partition function, we can obtain a matrix model for $U(N)$ $\mathcal{N}=2$ instanton partition function. But the matrix models obtained by using this method have too many matrices and are very difficult to deal with.

As in [6], in addition to inserting the identity operator, there is also another way to obtain matrix model from BPS partition functions, namely the non-intersecting path method introduced in [14, 15, 16, 6]. It would be interesting to see how to reproduce the matrix models presented in this paper by using non-intersecting path method and how to use this method to get matrix models of refined topological vertex. We hope that the investigation on the relationship between the non-intersecting paths and the refined topological vertex will deepen our understanding on the refined topological vertex and refined BPS partition function. This work is in progress.

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A. Refined matrix model for different chambers of the resolved conifold

A.1 The chamber $(R>0, 0<n<B<n+1)$

The refined partition function for chamber $(R>0, 0<n<B<n+1)$ of the resolved conifold is

$$\begin{aligned} Z_{BPS}^{ref}\Big|_{(R>0,0<n<B<n+1)} = &\left\langle 0 \left| \prod_{k=1}^{\infty} \Gamma_+^s \left(q_1^{k-\frac{1}{2}-\delta_1} q_2^{-\delta_2} (-Q)^{\frac{1}{2}} \right) \Gamma_+^{-s} \left(q_1^{k+n-\frac{1}{2}-\delta_1} q_2^{-\delta_2} (-Q)^{-\frac{1}{2}} \right) \right.\right. \\ &\bullet \prod_{l=1}^{n} \Gamma_-^s \left(q_2^{l-\frac{1}{2}+\delta_2} q_1^{\delta_1} (-Q)^{-\frac{1}{2}} \right) \Gamma_+^{-s} \left(q_1^{n-l+\frac{1}{2}-\delta_1} q_2^{-\delta_2} (-Q)^{-\frac{1}{2}} \right) \\ &\left.\left. \bullet \prod_{k=1}^{\infty} \Gamma_-^s \left(q_2^{k+n-\frac{1}{2}+\delta_2} q_1^{\delta_1} (-Q)^{-\frac{1}{2}} \right) \Gamma_-^{-s} \left(q_2^{k-\frac{1}{2}+\delta_2} q_1^{\delta_1} (-Q)^{\frac{1}{2}} \right) \right| 0 \right\rangle . \end{aligned} \quad \text{(A.1)}$$

We use the commutation relation of Γ_+^{-s} and Γ_-^s and obtain

$$\begin{aligned}
Z_{BPS}^{ref}\Big|_{(R>0,0<n<B<n+1)} &= \langle 0| \prod_{i=1}^{\infty} \Gamma_+^s \left(q_1^{i-\frac{1}{2}-\delta_1} q_2^{-\delta_2} (-Q)^{\frac{1}{2}} \right) \Gamma_+^{-s} \left(q_1^{i-\frac{1}{2}-\delta_1} q_2^{-\delta_2} (-Q)^{-\frac{1}{2}} \right) \\
&\bullet \prod_{j=1}^{\infty} \Gamma_-^s \left(q_2^{j-\frac{1}{2}+\delta_2} q_1^{\delta_1} (-Q)^{-\frac{1}{2}} \right) \Gamma_-^{-s} \left(q_2^{j-\frac{1}{2}+\delta_2} q_1^{\delta_1} (-Q)^{\frac{1}{2}} \right) |0\rangle \\
&\bullet \prod_{l=1}^{n} \prod_{p=1}^{l} \left(1 - Q^{-1} q_1^{n-l+\frac{1}{2}} \; q_2^{p-\frac{1}{2}} \right)^{-1} \qquad \text{(A.2)}
\end{aligned}$$

Then we insert the matrix model identity operator $\mathbb{I}$. Thus the corresponding matrix model is

$$\begin{aligned}
Z_{BPS}^{ref}\Big|_{(R>0,0<n<B<n+1)} &= \int dU \prod_{l=1}^{n} \prod_{p=1}^{l} \left(1 - Q^{-1} q_1^{n-l+\frac{1}{2}} \; q_2^{p-\frac{1}{2}} \right)^{-1} \qquad \text{(A.3)} \\
\times \left\langle 0 \middle| \prod_{l=1}^{\infty} \Gamma_+^s \left(q_1^{l-\frac{1}{2}-\delta_1} q_2^{-\delta_2} (-Q)^{\frac{1}{2}} \right) \Gamma_+^{-s} \left(q_1^{l-\frac{1}{2}-\delta_1} q_2^{-\delta_2} (-Q)^{-\frac{1}{2}} \right) \prod_{i=1}^{\infty} \Gamma'_-(u_i) \middle| 0 \right\rangle & \\
\times \left\langle 0 \middle| \prod_{j=1}^{\infty} \Gamma'_+ \left(u_j^{-1} \right) \prod_{r=1}^{\infty} \Gamma_-^s \left(q_2^{r-\frac{1}{2}+\delta_2} q_1^{\delta_1} (-Q)^{-\frac{1}{2}} \right) \Gamma_-^{-s} \left(q_2^{r-\frac{1}{2}+\delta_2} q_1^{\delta_1} (-Q)^{\frac{1}{2}} \right) \middle| 0 \right\rangle . &
\end{aligned}$$

Hence according to the equation (3.21, 3.22), the partition function of the matrix model is

$$\begin{aligned}
Z_{BPS}^{ref}\Big|_{(R>0,0<n<B<n+1)} &= \int dU \prod_{l=1}^{n} \prod_{p=1}^{l} \left(1 - Q^{-1} q_1^{n-l+\frac{1}{2}} q_2^{p-\frac{1}{2}} \right)^{-1} \\
\bullet \mathrm{Det}^s & \left[\prod_{k=0}^{\infty} \frac{\left(1 + sU^{-1} (-Q)^{-\frac{1}{2}} q_2^{k-\frac{1}{2}+\delta_2} q_1^{\delta_1} \right) \left(1 + sU (-Q)^{\frac{1}{2}} q_1^{k-\frac{1}{2}-\delta_1} q_2^{-\delta_2} \right)}{\left(1 - sU^{-1} (-Q)^{\frac{1}{2}} q_2^{k-\frac{1}{2}+\delta_2} q_1^{\delta_1} \right) \left(1 - sU (-Q)^{-\frac{1}{2}} q_1^{k-\frac{1}{2}-\delta_1} q_2^{-\delta_2} \right)} \right] . \qquad \text{(A.4)}
\end{aligned}$$

A.2 The chamber $(R > 0, n-1 < B < n \leq 0)$

The refined partition function for chamber $(R > 0, n-1 < B < n \leq 0)$ of the resolved conifold is

$$\begin{aligned}
Z_{BPS}^{ref}\Big|_{(R>0,n-1<B<n\leq 0)} &= \left\langle 0 \middle| \prod_{k=1}^{\infty} \Gamma_+^s \left(q_1^{k-\frac{1}{2}-\delta_1} q_2^{-\delta_2} (-Q)^{-\frac{1}{2}} \right) \Gamma_+^{-s} \left(q_1^{k+n-\frac{1}{2}-\delta_1} q_2^{-\delta_2} (-Q)^{\frac{1}{2}} \right) \right. \\
&\bullet \prod_{l=1}^{n} \Gamma_-^s \left(q_2^{l-\frac{1}{2}+\delta_2} q_1^{\delta_1} (-Q)^{\frac{1}{2}} \right) \Gamma_+^{-s} \left(q_1^{n-l+\frac{1}{2}-\delta_1} q_2^{-\delta_2} (-Q)^{\frac{1}{2}} \right) \\
&\bullet \left. \prod_{k=1}^{\infty} \Gamma_-^s \left(q_2^{k+n-\frac{1}{2}+\delta_2} q_1^{\delta_1} (-Q)^{\frac{1}{2}} \right) \Gamma_-^{-s} \left(q_2^{k-\frac{1}{2}+\delta_2} q_1^{\delta_1} (-Q)^{-\frac{1}{2}} \right) \middle| 0 \right\rangle .
\end{aligned}$$

(A.5)

We use the commutation relation of Γ_+^{-s} and Γ_-^s and obtain

$$
\begin{aligned}
Z_{BPS}^{ref}\Big|_{(R>0,n-1<B<n\leq 0)} &= \langle 0| \prod_{i=1}^{\infty} \Gamma_+^s \left(q_1^{i-\frac{1}{2}-\delta_1} q_2^{-\delta_2} (-Q)^{-\frac{1}{2}} \right) \Gamma_+^{-s} \left(q_1^{i-\frac{1}{2}-\delta_1} q_2^{-\delta_2} (-Q)^{\frac{1}{2}} \right) \\
&\bullet \prod_{j=1}^{\infty} \Gamma_-^s \left(q_2^{j-\frac{1}{2}+\delta_2} q_1^{\delta_1} (-Q)^{\frac{1}{2}} \right) \Gamma_-^{-s} \left(q_2^{j-\frac{1}{2}+\delta_2} q_1^{\delta_1} (-Q)^{-\frac{1}{2}} \right) |0\rangle \\
&\bullet \prod_{l=1}^{n} \prod_{p=1}^{l} \left(1 - Q q_1^{n-l+\frac{1}{2}} \; q_2^{p-\frac{1}{2}} \right)^{-1} \qquad \text{(A.6)}
\end{aligned}
$$

Then we insert the matrix model identity operator $\mathbb{I}$. Thus the corresponding matrix model is

$$
\begin{aligned}
Z_{BPS}^{ref}\Big|_{(R>0,n-1<B<n\leq 0)} &= \int dU \prod_{l=1}^{n} \prod_{p=1}^{l} \left(1 - Q q_1^{n-l+\frac{1}{2}} \; q_2^{p-\frac{1}{2}} \right)^{-1} \qquad \text{(A.7)} \\
\times \left\langle 0 \left| \prod_{l=1}^{\infty} \Gamma_+^s \left(q_1^{l-\frac{1}{2}-\delta_1} q_2^{-\delta_2} (-Q)^{-\frac{1}{2}} \right) \Gamma_+^{-s} \left(q_1^{l-\frac{1}{2}-\delta_1} q_2^{-\delta_2} (-Q)^{\frac{1}{2}} \right) \prod_{i=1}^{\infty} \Gamma_-'(u_i) \right| 0 \right\rangle \\
\times \left\langle 0 \left| \prod_{j=1}^{\infty} \Gamma_+' \left(u_j^{-1} \right) \prod_{r=1}^{\infty} \Gamma_-^s \left(q_2^{r-\frac{1}{2}+\delta_2} q_1^{\delta_1} (-Q)^{\frac{1}{2}} \right) \Gamma_-^{-s} \left(q_2^{r-\frac{1}{2}+\delta_2} q_1^{\delta_1} (-Q)^{-\frac{1}{2}} \right) \right| 0 \right\rangle .
\end{aligned}
$$

Hence according to the equation (3.21, 3.22), the partition function of the matrix model is

$$
\begin{aligned}
Z_{BPS}^{ref}\Big|_{(R>0,n-1<B<n\leq 0)} &= \int dU \prod_{l=1}^{n} \prod_{p=1}^{l} \left(1 - Q q_1^{n-l+\frac{1}{2}} q_2^{p-\frac{1}{2}} \right)^{-1} \\
\bullet \mathrm{Det}^s \left[\prod_{k=0}^{\infty} \frac{\left(1 + sU^{-1} (-Q)^{\frac{1}{2}} q_2^{r-\frac{1}{2}+\delta_2} q_1^{\delta_1} \right) \left(1 + sU (-Q)^{-\frac{1}{2}} q_1^{l-\frac{1}{2}-\delta_1} q_2^{-\delta_2} \right)}{\left(1 - sU^{-1} (-Q)^{-\frac{1}{2}} q_2^{r-\frac{1}{2}+\delta_2} q_1^{\delta_1} \right) \left(1 - sU (-Q)^{\frac{1}{2}} q_1^{l-\frac{1}{2}-\delta_1} q_2^{-\delta_2} \right)} \right] &. \qquad \text{(A.8)}
\end{aligned}
$$

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